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On 2-normed spaces

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First we shall recall some definitions needed in the squel.

Let X be a set. Consider a mapping $\rho : X \times X \times X \to R$. ρ is called a *2-metric* (S. Gähler), if it satisfies

(1) $\rho(x, y, z) \neq 0$ for any $x, y (x \neq y)$ and some z,

(2) $\rho(x, y, z) = 0$, if at least two points of therr points x, y, z are equal,

(3) ρ is symmetric on x, y, z, i. e. $\rho(x, y, z) = \rho(x, y, z) = \dots = \rho(z, y, x)$,

(4) $\rho(x, y, z) \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$.

By a *(linear) 2-normed space* X *over reals* (S. Gähler), we mean a linear space X in which to each pair of points $x, y \in X$, there exists a real number $||x, y||$ satisfying the following properties

(1) $||x, y|| = 0 \Leftrightarrow x, y$ are linear dependent,

(2) $||x, y|| = ||y, x||$,

(3) $||\alpha x, y|| = |\alpha|\, ||x, y||$ for any real α,

(4) $||x, y + z|| \leq ||x, y|| + ||x, z||$.

We assume $\dim L \geq 2$.

On a 2-normed space X,

$$\rho(x, y, z) = ||y - x, z - x||$$

defines a 2-metric on X.

A *2-inner product space* (R. Ehret) is a linear space X with a mapping $(\cdot, \cdot|\cdot) : X \times X \times X \to R$ satisfying the following conditions:

(1) $(a, a|b) \geq 0$, and $(a, a|b) = 0 \Leftrightarrow a, b$ are linearly dependent,

(2) $(a, a|b) = (b, a|a)$,

(3) $(a, b|c) = (b, a|c)$,

(4) $(\alpha a, b|c) = \alpha(a, b|c)$ for any real α,

(5) $(a + a', b|c) = (a, b|c) + (a', b|c)$.

Proposition 1. On a 2-inner product space,

$$\|a, b\| = \sqrt{(a, a|b)}$$

define a 2-norm for which

$$(a, b|c) = \frac{1}{4}(\|a + b, c\|^2 - \|a - b, c\|^2)$$

and

$$\|a + b, c\|^2 + \|a - b, c\|^2 = 2(\|a, c\|^2 + \|b, c\|^2).$$

Proposition 2. Let X be a pre-Hilbert space. Then

$$(a, b|c) = \begin{vmatrix} (a, b) & (a, c) \\ (b, c) & (c, c) \end{vmatrix}$$

$$= (a, b)\|c\|^2 - (a, c)(b, c)$$

defines a 2-inner product on X.

2-normed space X is called *strictly convex* if for $a, b = 0$, $\|a + b, c\| = \|a, c\| + \|b, c\|$ and $\|a, c\| = \|b, c\| = 1$, where c is linearly independent to a, b, implies $a = b$, equivalently $\|a, c\| = \|b, c\| = \frac{1}{2}\|a + b, c\| = 1$, where c is linear independent to a, b, implies $a = b$.

A 2-normed space X is said to be *strictly 2-convex*, if $\|a, b\| = \|a, c\| = \|b, c\| = \frac{1}{3}\|a + c, b + c\| = 1$ implies $c = a + b$.

Let c be a fixed non-zero element of a 2-normed space X, and let $V(c)$ be the linear subspace of X generated by c.

Then we obtain the quotient space $X/V(c) = L_c$. We put

$$||x||_c = ||x, c||,$$

then $||\cdot||_c$ is well-defined on X_c.

Proposition 3. $||\cdot||_c$ is a norm on X_c.

Proposition 4. A 2-normed space X is strictly convex if and only if $X_c (c \neq 0)$ is strictly convex in usual sense, i. e. $||x + y||_c = ||X||_c + ||y||_c$, $||X||_c = ||y||_c = 1$ imply $x = y$.

Some new characterizations of the strictly convexity by bounded linear 2-functionals in some sense are recently given by Y. Cho, K. Ha and W, Kim [1].

Quite recently, a wonderful characterization of a 2-inner product space is given by C. Diminnie and A. White [2].

Proposition 5. 2-normed space is a 2-inner product space if and only if

$$||x + y, y + z||^2 + ||x + y, y - z||^2$$
$$+ ||x - y, y + z||^2 + ||x - y, y - z||^2$$
$$= ||x, y||^2 + ||x, z||^2 + ||y, z||^2$$

holds.

Next we concern with some special classes of mappings.

A 2-metric space X is called to be complete, if for any sequence $\{x_n\}$, $\rho(x_m, x_n, a) \to 0$ for all $a \in X (m, n \to \infty)$ implies $\rho(x_m, x, a) \to 0$ for all $a \in X$ and for some $x \in X (m \to \infty)$.

The following fixed point theorem is obtained.

Proposition 6. Let X be a bounded complete 2-metric space, and let f_n be a sequence of mappings of X into itself. If there are non-negative α, β such that for all $x, y \in X$

$$\rho(f_m(x), f_n(y), a) \leq \alpha(\rho(x, f_m(x), a)$$
$$+ \rho(y, f_n(y), a)) + \beta\rho(x, y, a)$$

with $2\alpha + \beta < 1$, then the sequence $\{f_n\}$ has a unique common fixed point (K. Iséki, P. L. Sharma and B. K. Sharma).

Let E be a usual normed space, and let X be a 2-normed space. Then the following three mappings are considered.

(1) $f_1 : E \to X$ satisfies

$$||f(x) - f(y)|| = ||x - y, c||$$

for some fixed $c \in X$.

(2) $f_2 : X \to E$ satisfies

$$||f(x) - f(y), c|| = ||x - y||$$

for some fixed $c \in X$.

(3) $f_3 : X \to X$ satisfies

$$||f(x) - f(y), c|| \leq ||x - y, c||$$

for some fixed $c \in X$.

The first two mappings f_1, f_2 are due to C. Diminnie and A. White. f_3 is discussed by the present author. The mapping f_3 is uniquely determined. Roughly speaking this type is of an affine mapping (Iséki-Diminnie-White).

References

[1] Y. J. Cho, K. S. Ha and W. S. Kim, Strictly convex linear 2-normed spaces, Math. Japonica, 26(1981), 475-478.

[2] C. Diminnie and A. White, A characterization of 2-inner product spaces, to appear in Math. Japonica.